UDC 539.3

GENERALIZED PERIODIC PROBLEM OF THE THEORY OF ELASTICITY

PMM Vol.42, № 3, 1978, pp. 521-531

M.L.BURYSHKIN
(Odessa)

(Received February 15,1977)

The ordinary periodic problem for an isotropic medium has been studied in sufficient detail in [1]. In spite of the infinite connectivity, the periodic stress-strain state can be easily investigated, since it reduces to the problem of determining two functions, analytic on the outside of the basic hole [2]. The periodic states however do not exhaust the variety of the practical problems dealing with the distribution of stresses in a medium weakened by a regular series of holes.

Use of the group representation theory methods extends significantly the class of loads which allow an effective analysis of the periodic stress-strain state of an elastic medium. Instead of the condition of periodicity of the load function, it is demanded that the function transforms according to some unspecified representation of a symmetry group. It is shown that the corresponding problem of the theory of elasticity can be reduced to that of finding four functions analytic on the outside of the basic hole. The class of the functions under consideration is very general, consequently many loads interesting from the engineer's point of view can be represented in the form of a linear finite combination of the components transformable in terms of the irreducible representations. The theoretical basis for all this is provided in [3]. The basic results are illustrated by several specific examples. The approach utilized can be suitably extended to become applicable to the cyclic and doubly periodic problems (with differing lattices) of the theory of elasticity.

1. Basic concepts. We study a biharmonic problem for an isotropic medium weakened by a series of holes and possessing group C_s (C_{2v}^1 or D_{2h}^1) symmetry. The elements of this group are translations (shifts) T_m along the x -axis onto the segments $2ml$, and reflections θ_m $(m = 0, \pm 1, \pm 2, \ldots)$ in the planes $\Pi_m = T_{m/2}\Pi_0$ (Fig. 1). The hole contours are under the load $Q_{\alpha\nu\mu}$ $(\mu = 1, 2, \ldots, m_\alpha)$ which transforms according to the irreducible representation $\tau_{\alpha\nu}$ of the group C_s of dimension m_α [3]. To make it clearer, the general part of this paper uses the formulation of the plane problem of the theory of elasticity.

We say of each function $p_{\alpha\nu\mu}$ $(\mu = 1, 2, \ldots, m_\alpha)$ that it transforms according to the irreducible representation $\tau_{\alpha\nu}$ if the following relations hold in the invariant coordinate system:

$$p_{\alpha\nu\mu}(gz) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\, p_{\alpha\nu\rho}(z), \quad \forall g \in C_s, \quad \forall z \in E \tag{1.1}$$

Here $\tau_{\alpha\nu\mu\rho}(g)$ is the $\mu\rho$ -th element of the matrix $\tau_{\alpha\nu}(g)$ of the representation $\tau_{\alpha\nu}$ and E is the domain of definition of the functions $p_{\alpha\nu\mu}$, possessing the group C_s symmetry. In the present case E is the domain of a complex plane and z is its representative point.

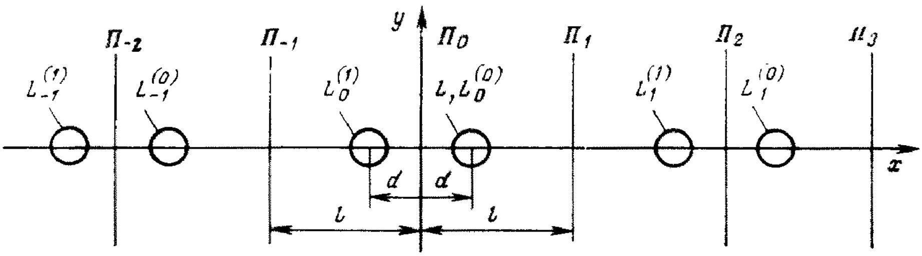

Fig. 1

According to [3], the stress-strain state under the load $Q_{\alpha\nu\mu}$ will be transformed in the same manner according to the representation $\tau_{\alpha\nu}$, i. e. any components of this state written in terms of the invariant reference system will satisfy the relations (1. 1) (*). If a load $Q_{\alpha\nu\mu}$ acting in the medium occupying the domain E generates normal ($\sigma_x^{(\mu)}$, $\sigma_y^{(\mu)}$) and tangential ($\tau_{xy}^{(\mu)}$) stresses, then according to the statements given above we have

$$\sigma_{gx}^{(\mu)}(gz) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\,\sigma_x^{(\rho)}(z), \quad \sigma_{gy}^{(\mu)}(gz) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\,\sigma_y^{(\rho)}(z) \tag{1.2}$$

$$\tau_{gx,\,gy}^{(\mu)}(gz) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\,\tau_{xy}^{(\rho)}(z), \quad \forall g \in C_s, \quad \forall z \in E \quad (\mu = 1, 2, \ldots, m_\alpha)$$

In what follows, we shall denote the basic contour by L or $L_0^{(0)}$, $L_0^{(1)} = \theta_0 L_0^{(0)}$, $L_m^{(j)} = T_m L_0^{(j)}$ (Fig. 1) and introduce the following notation: $t_m^{(j)} \in L_m^{(j)}$ is a point of the contour $L_m^{(j)}$, $t = t_0^{(0)}$, d is the distance from the characteristic point of the basic hole (in the case of a circular hole this would be its center) to the imaginary axis $\varphi_{\alpha\nu\mu}(z)$ and $\psi^{*}_{\alpha\nu\mu}(z)$ are the complex Kolosov — Mushelishvili functions describing the stress-strain state of the medium under the load $Q_{\alpha\nu\mu}$, and $\psi_{\alpha\nu\mu}(z) = \psi^{*}_{\alpha\nu\mu}(z) + z\varphi'_{\alpha\nu\mu}(z)$ is the Sherman function. Here and in the following $j = 0, 1$; $m = 0, \pm 1, \pm 2, \ldots$; $\mu, \rho = 1, 2, \ldots, m_\alpha$

2. Properties of the complex functions. By virtue of the known [4] relations connecting the combinations of the functions $\varphi(z)$ and $\psi(z)$ with the stresses, the relations (1. 2) assume the form

$$[\varphi'_{\alpha\nu\mu}(z) + \overline{\varphi'_{\alpha\mu\nu}(z)}]_{gx,\,gy} = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\,[\varphi'_{\alpha\nu\rho}(z) + \overline{\varphi'_{\alpha\nu\rho}(z)}] \tag{2.1}$$

$$[(\bar{z} - z)\,\varphi''_{\alpha\nu\mu}(z) - \varphi'_{\alpha\nu\mu}(z) + \psi'_{\alpha\nu\mu}(z)]_{gx,\,gy} = \tag{2.2}$$

$$\sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\,[(\bar{z} - z)\,\varphi''_{\alpha\nu\rho}(z) - \varphi'_{\alpha\nu\rho}(z) + \psi'_{\alpha\nu\rho}(z)]$$

*) This fact and a series of concepts connected with the elementary cell method were explained in more detail in the manuscript deposited by the author and entitled " On the application of the theory of representation of discrete groups in the problem of equilibrium and small oscillations of linearly elastic systems. VINITI, No. 208-75, 1975.

In the δ -neighborhood of some point $z_0 \in E$ the functions $\varphi_{\alpha\nu\mu}(z)$ and $\psi_{\alpha\nu\mu}(z)$ are analytic, i.e.

$$\varphi_{\alpha\nu\mu}(z) = \sum_{k=0}^{\infty} a_{\mu k}(z - z_0)^k, \quad \psi_{\alpha\nu\mu}(z) = \sum_{k=0}^{\infty} b_{\mu k}(z - z_0)^k, \ \forall z \in \delta \tag{2.3}$$

and the symmetry of the domain E implies

$$\varphi_{\alpha\nu\mu}(gz) = \sum_{k=0}^{\infty} a_{\mu k}^{(1)}(gz - gz_0)^k, \quad \psi_{\alpha\nu\mu}(gz) = \sum_{k=0}^{\infty} b_{\mu k}^{(1)}(gz - gz_0)^k, \ \forall z \in \delta \tag{2.4}$$

where $a_{\mu k}$, $b_{\mu k}$, $a_{\mu k}{}^{(1)}$, $b_{\mu k}{}^{(1)}$ $(k = 0, 1, 2, \ldots)$ are complex coefficients.

Since the function $\mathrm{Re}\,\varphi'(z)$ is invariant with respect to the coordinate system, the expression (2.1) simplifies to

$$\varphi'_{\alpha\nu\mu}(gz) + \overline{\varphi'_{\alpha\nu\mu}(gz)} = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(g)\,[\varphi'_{\alpha\nu\rho}(z) + \overline{\varphi'_{\alpha\nu\rho}(z)}] \tag{2.5}$$

Let initially $g = T_m$ and hence $gz = z + 2ml$. Substituting the series (2.3) and (2.4) into (2.5) and equating the coefficients of like powers of $(z - z_0)^k$, we obtain

$$a_{\mu k}^{(1)} = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(T_m)\, a_{\rho k} \quad (k = 1, 2, \ldots)$$

Then from (2.3) and (2.4) it follows

$$\varphi_{\alpha\nu\mu}(z + 2ml) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(T_m) \sum_{k=0}^{\infty} a_{\rho k}(z - z_0)^k = \tag{2.6}$$
$$\sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(T_m)\, \varphi_{\alpha\nu\rho}(z)$$

Taking into account the property (2.6) we find from (2.2), that

$$\psi_{\alpha\nu\mu}(z + 2ml) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(T_m)\, \psi_{\alpha\nu\rho}(z) \tag{2.7}$$

Let now $g = \theta_m$ and $gz = -\bar{z} + 2ml$. Carrying out the procedure given above we find that

$$a_{\mu k}^{(1)} = (-1)^{k-1} \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(\theta_m)\, \overline{a_{\rho k}}$$

and

$$\varphi_{\alpha\nu\mu}(-\bar{z} + 2ml) = \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(\theta_m) \sum_{k=0}^{\infty} (-1)^{k-1}\, \overline{a_{\rho k}}(-\bar{z} + \bar{z}_0)^k = \tag{2.8}$$
$$- \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(\theta_m)\, \overline{\varphi_{\alpha\nu\rho}(z)}$$

We use the relation (2.8) to establish the validity of

$$[(\bar{z} - z)\,\varphi''_{\alpha\nu\mu}(z) - \varphi'_{\alpha\nu\mu}(z) + \psi'_{\alpha\nu\mu}(z)]_{\theta_m x,\, \theta_m y} =$$

$$\frac{(\bar{z}-z)\sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(\theta_m)\,\varphi''_{\alpha\nu\rho}(z) - \sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(\theta_m)\,\varphi'_{\alpha\nu\rho}(z) +}{\psi'_{\alpha\nu\mu}(-\bar{z}+2ml)}$$

and then find, that from (2.2) it follows

$$\psi_{\alpha\nu\mu}(-\bar{z}+2ml) = -\sum_{\rho=1}^{m_\alpha} \tau_{\alpha\nu\mu\rho}(\theta_m)\,\overline{\psi_{\alpha\nu\rho}(z)} \tag{2.9}$$

3. General form of the complex functions. Assuming that the functions $\varphi_{\alpha\nu\mu}(z)$ are holomorphic in the domain occupied by the medium and vanish at infinity, we can write them, in accordance with [5], in the Cauchy integral form

$$\varphi_{\alpha\nu\mu}(z) = \lim_{N\to\infty} \sum_{m=-N}^{N} \sum_{j=1}^{2} I_{\mu m}^{(j)}(z), \quad I_{\mu m}^{(j)}(z) = \frac{1}{2\pi i}\int\limits_{L_m^{(j)}} \frac{\varphi_{\alpha\nu\mu}(t_m^{(j)})\,dt_m^{(j)}}{t_m^{(j)}-z}$$

Taking into account (2.6), (2.8) and the fact that the matrix $\tau_{\alpha\nu}(g)$ is unitary and integrating along the basic contour, we obtain

$$\varphi_{\alpha\nu\mu}(z) = \sum_{\rho=1}^{m_\alpha} \lim_{N\to\infty} \sum_{m=-N}^{N} [\tau_{\alpha\nu\rho\mu}(T_m)\,\Phi^{(\rho)}(z+2ml) - \tag{3.1}$$
$$\tau_{\alpha\nu\rho\mu}(\theta_m)\,\overline{\Phi^{(\rho)}(-\bar{z}+2ml)}]$$
$$\Phi^{(\rho)}(z) = I_{\rho 0}^{(0)}(z) = \frac{1}{2\pi i}\int\limits_{L} \frac{\varphi_{\alpha\nu\rho}(t)\,dt}{t-z}$$

Similarly, using the properties (2.7) and (2.9) we obtain

$$\psi_{\alpha\nu\mu}(z) = \sum_{\rho=1}^{m_\alpha} \lim_{N\to\infty} \sum_{m=-N}^{N} [\tau_{\alpha\nu\rho\mu}(T_m)\,\Psi^{(\rho)}(z+2ml) - \tag{3.2}$$
$$\tau_{\alpha\nu\rho\mu}(\theta_m)\,\overline{\Psi^{(\rho)}(-\bar{z}+2ml)}]$$

The functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ in (3.1) and (3.2) are holomorphic outside the basic contour. Using the elementary properties of group representation we can show, that for any functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ analytic in this region the formulas given define the functions $\varphi_{\alpha\nu\mu}(z)$ and $\psi_{\alpha\nu\mu}(z)$ with properties (2.6) – (2.9). This enables us to assert that the expressions (3.1) and (3.2) hold for the complex potentials $\varphi_{\alpha\nu\mu}(z)$ and $\psi_{\alpha\nu\mu}(z)$ describing any stress-strain state of the medium transformable according to the irreducible representation $\tau_{\alpha\nu}$ of the group C_s.

4. Converse problem of the theory of elasticity for narrow compressed strips. The problem given here serves as an elementary illustration of application of the formulas (3.1) and (3.2) corresponding to the one- dimensional representations $\tau_{\alpha\nu}$ ($\alpha = 0, \pi$; $\nu = 1, 2$) of the group C_s. We determine the form of the opening of uniform strength in a strip, with homogeneous stress-strain state: $\sigma_x^{(0)} = p$, $\sigma_y^{(0)} = q$, $\tau_{xy}^{(0)} = 0$. We assume that the opening is situated near the edge, and that the edge effect exerts an appreciable influence on the form of the opening.

Normal stresses of intensity P should be present at the contour.

The solution is obtained using an approximate, though simple method of a small parameter. Use of the formulas (3.1) and (3.2) makes it possible to solve the problem effectively using more powerful methods [6]. Within the strip in question bounded by the planes Π_0 and Π_1 (Fig. 1), we have

$$\sigma_x(z) = p + \sigma_x^{(1)}(z), \quad \sigma_y(z) = q + \sigma_y^{(1)}(z), \quad \tau_{xy}(z) = \tau_{xy}^{(1)}(z) \qquad (4.1)$$

where the superscript $^{(1)}$ denotes the stresses caused by the contour load $Q_1(t)$ for which $\sigma_r^{(1)}(t) = P - \sigma_r^{(0)}(t)$ and $\tau_{r\theta}^{(1)}(t) = -\tau_{r\theta}^{(0)}(t)$.

We investigate two types of the boundary conditions on the rectilinear edges: (a) free support where the points of the edge may move in the y-direction only, and (b) rigid coupling where only a motion in the x- direction is possible. Combining these conditions in different ways, we arrive at four possible types of strips $S_{\alpha\nu}$ ($\alpha = 0, \pi$; $\nu = 1, 2$) which represent elementary systems in the sense of one-dimensional irreducible representation $\tau_{\alpha\nu}$. (*), with respect to the infinite plate S with group C_s symmetry. The author states in [3] that the stress-strain state of the strip $S_{\alpha\nu}$ under a contour load $Q_1(t)$ is identical with the state of the corresponding cell of the plate S under a load obtained by continuation of the function $Q_1(t)$ from the basic contour to the whole plate in accordance with the irreducible representation $\tau_{\alpha\nu}$, i.e. by using the formula (1.1). In this case the boundary conditions at the strip edges are satisfied automatically.

From the known relations of the theory of elasticity [7] and (4.1), we have

$$\sigma_r(z) + \sigma_\theta(z) = p + q + 4\,\mathrm{Re}\,\varphi'_{\alpha\nu 1}(z),\ \sigma_\theta(z) - \sigma_r,\ \sigma_\theta(z)(z) + 2i\tau_{r\theta}(z) =$$
$$\sigma^2 \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \{q - p + 2[(\bar z - z)\varphi''_{\alpha\nu 1}(z) - \varphi'_{\alpha\nu 1}(z) + \psi'_{\alpha\nu 1}(z)]\},\quad \sigma = e^{i\theta}$$

where θ denotes the polar angle. Taking into account the condition of uniform strength $\sigma_\theta(t) = A = \mathrm{const}$ and the boundary conditions on the contour L, we can replace the last formulas by

$$4\,\mathrm{Re}\,\varphi_{\alpha\nu 1}(t) = P + A - p - q \qquad (4.2)$$

$$\sigma^2 \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \{q - p + 2[(\bar t - t)\varphi''_{\alpha\nu 1}(t) - \varphi'_{\alpha\nu 1}(t) + \psi'_{\alpha\nu 1}(t)]\} = A - P \qquad (4.3)$$

It should be assumed here that the function $z_* = d + \omega(\zeta)$ maps conformally the outside of the unit circle of the complex ζ -plane onto the outside of the uniform strength contour which is to be determined.

The relations (4.2) together with the usual arguments [7] yield

$$\varphi_{\alpha\nu 1}(z) = 0, \quad A = p + q - P$$

*) Buryshkin M.L., Romanenko F.A. and Sheianova E.N. Stress concentration around a circular hole in a strip of finite width. Theses of lectures given at the All-Union conference " Perfecting the Methods of Computing and Design of Buildings and Structures Erected in Seismically Active Zones", pt. 3, Kishinev, 1976.

and this converts (4.3) to the form

$$\sigma^2 \frac{\omega'(\sigma)}{\overline{\omega'(\sigma)}} \left\{\frac{q-p}{2} + \psi'_{\alpha\nu 1}(t)\right\} = \frac{p+q}{2} - P \tag{4.4}$$

Assuming now

$$\omega(\zeta) = \zeta + \sum_{s=1}^{\infty} v_s \zeta^{-s}, \quad \Psi(\zeta) = \Psi^{(1)'}[d + \omega(\zeta)] = \sum_{k=2}^{\infty} b_k \zeta^{-k}$$

$$\psi(\zeta) = \psi'_{\alpha\nu 1}[d + \omega(\zeta)], \quad \varepsilon = \frac{1}{2l}, \quad \varepsilon_1 = \frac{d}{l}$$

and neglecting the powers of the small parameter ε greater than third we find from (3.2), as in [7], that

$$\psi(\zeta) = \sum_{k=2}^{\infty} b_k \zeta^{-k} + \varepsilon^2 b_2 \left(\lambda_2^{(1)(1)} + \lambda_2^{(11)(2)}\right) + \varepsilon^3 b_3 \left(\lambda_3^{(11)(2)} - \lambda_3^{(11)(1)}\right) + 2\varepsilon^3 b_2 \left(\lambda_3^{(11)(1)} + \lambda_3^{(11)(2)}\right) \omega(\zeta) \tag{4.5}$$

Here and henceforth

$$\lambda_p^{(\rho\mu)(1)} = \lim_{N\to\infty} \sum_{m=-N}^{N}{}^{*} \frac{\tau_{\alpha\nu\rho\mu}(T_m)}{m^p}, \quad \lambda_p^{(\rho\mu)(2)} = \lim_{N\to\infty} \sum_{m=-N}^{N} \frac{\tau_{\alpha\nu\rho\mu}(\theta_m)}{(m-\varepsilon_1)^p} \tag{4.6}$$

and the asterisk denotes the absence of the term corresponding to the value $m = 0$.

Substituting (4.5) into the condition (4.4), passing to a system of algebraic equations for the coefficients v_s $(s = 1, 2, \ldots)$ and b_k $(k = 2, 3, \ldots)$ and solving the latter with the help of the method of a small paramater, we find that

$$\omega(\zeta) = \zeta + \frac{B_0 - \left(\lambda_2^{(11)(1)} + \lambda_2^{(11)(2)}\right) B_1 \varepsilon^2}{\zeta} - \frac{\left(\lambda_3^{(11)(1)} + \lambda_3^{(11)(2)}\right) B_1 \varepsilon^3}{\zeta^2} \tag{4.7}$$

$$B_0 = \frac{p-q}{p+q-2P}, \quad B_1 = 4\frac{pq - P(p+q) + P^2}{(p+q-2P)^2}$$

The indices in the expression for $S_{\alpha\nu}$ $(\alpha = 0, \pi;\ \nu = 1, 2)$ and, consequently, in the formula (4.6), are chosen accroding to the variant of the boundary conditions of the strip used (*). In the case of strips of double width (with the boundary conditions of the type S_{01} and $S_{\pi 1}$) weakened by two holes symmetrically distributed about the plane Π_0, a method of loading, symmetric or skew symmetric with respect to this plane, does not result in new forms of equally strong contours. In a particular case of $\alpha = 0$, $\nu = 1$, $\varepsilon_1 = 1/2$, the problem becomes ordinary periodic, and the function (4.7) describes a form of an equal strength hole obtained in [7].

5. Generalized periodic problem of flexure of thin plates. The contour load in question of a plate with periodic structure is described by one of the functions $Q_{\alpha\nu\mu}$ $(\mu = 1, 2, \ldots, m_\alpha)$ transformable according to the irreducible representation $\tau_{\alpha\nu}$ of the group C_s. According to the formulas (3.1) and (3.2), the

*) see the last footnote.

computation of the stress-strain state of the plate reduces to determination of the complex functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ analytic outside the basic opening. The functions are obtained from the system of boundary conditions [8] on the contour L

$$K_1\varphi_{\alpha\nu\mu}(t) + K_2[(t-\bar{t})\overline{\varphi'_{\alpha\nu\mu}(t)} + \overline{\psi_{\alpha\nu\mu}(t)}] = f^{(\mu)}(t) + ic_\mu t \quad (\mu = 1, 2, \ldots, m_\alpha) \tag{5.1}$$

Here K_1 and K_2 denote the coefficients, c_μ is a real constant obtained from the consideration of uniqueness of the flexure, and $f^{(\mu)}(t)$ is a function depending on the load $Q_{\alpha\nu\mu}$. On the remaining contours the boundary conditions are satisfied automatically by virtue of the relations of the type (1.1) or (1.2). The simplifications associated with the generalized periodic problem consist of the fact that the dimension m_α of the representation $\tau_{\alpha\nu}$ is not greater than two. Such a problem represents a natural generalization of the usual periodic problem corresponding to a unique irreducible representation. It is expedient to remember that any method used to solve the last problem can be extended to a general case. We illustrate this below on the method of a small parameter [5].

Assuming that the functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ are holomorphic and vanish at infinity and writing them in the integral Cauchy form, we find, after manipulating the formulas (3.1) and (3.2), that in the neighborhood of the basic contour

$$\varphi_{\alpha\nu\mu}(z) = \Phi^{(\mu)}(z) - \sum_{k=0}^{\infty} \varepsilon^{k+1} \sum_{\rho=1}^{m_\alpha} [\lambda_{k+1}^{(\rho\mu)(1)} J_\Phi^{(\rho)}(z, k) - \lambda_{k+1}^{(\rho\mu)(2)} \overline{J_\Phi^{(\rho)}(-\bar{z}+2d, k)}] \tag{5.2}$$

$$\psi_{\alpha\nu\mu}(z) = \Psi^{(\mu)}(z) - \sum_{k=0}^{\infty} \varepsilon^{k+1} \sum_{\rho=1}^{m_\alpha} [\lambda_{k+1}^{(\rho\mu)(1)} J_\Psi^{(\rho)}(z, k) - \lambda_{k+1}^{(\rho\mu)(2)} \overline{J_\Psi^{(\rho)}(-\bar{z}+2d, k)}] \tag{5.3}$$

$$J_\Delta^{(\rho)}(z, k) = \frac{1}{2\pi i}\int_L \Delta^{(\rho)}(\xi)(\xi - z)^k d\xi \quad (\Delta = \Phi, \Psi)$$

where ξ is a point of the contour L.

The unknown functions $\Phi^{(\rho)}(z)$ and $\Psi^{(\rho)}(z)$ are sought in the form of series in powers of

$$\Phi^{(\rho)}(z) = \sum_{s=0}^{\infty} \varepsilon^s \Phi_s^{(\rho)}(z), \quad \Psi^{(\rho)}(z) = \sum_{s=0}^{\infty} \varepsilon^s \Psi_s^{(\rho)}(z) \tag{5.4}$$

Taking into account (5.4), we substitute the expressions (5.2) and (5.3) into the boundary condition (5.1). Equating the coefficients of like powers of small parameter ε in both sides of the resulting equation, yields the following infinite system of functional equations:

$$K_1\Phi_s^{(\mu)}(t) + K_2[(t-\bar{t})\overline{\Phi_s^{(\mu)\prime}(t)} + \overline{\Psi_s^{(\mu)}(t)}] = f_s^{(\mu)}(t) \quad (s = 0, 1, 2, \ldots) \tag{5.5}$$

where

$$f_0^{(\mu)}(t) = f^{(\mu)}(t) + ic_\mu t, \quad f_1^{(\mu)}(t) = \sum_{\rho=1}^{m_\alpha} \{K_1[\lambda_1^{(\rho\mu)(1)} J_{\Phi,0}^{(\rho)}(t, 0) -$$

$$\lambda_1^{(\rho\mu)(2)} \overline{J_{\Phi,0}^{(\rho)}(t,0)}] + K_2 [\lambda_1^{(\rho\mu)(1)} \overline{J_{\Psi,0}^{(\rho)}(t,0)} - \lambda_1^{(\rho\mu)(2)} J_{\Psi,0}^{(\rho)}(t,0)]\}$$

$$f_s^{(\mu)}(t) = \sum_{\rho=1}^{m_\alpha} \Big\{ \sum_{k=0}^{s-1} K_1 [\lambda_{k+1}^{(\rho\mu)(1)} J_{\Phi,s-k-1}^{(\rho)}(t,k) - \lambda_{k+1}^{(\rho\mu)(2)} \overline{J_{\Phi,s-k-1}^{(\rho)}(-\bar{t}+2d,k)}] -$$
$$K_2 (t-\bar{t}) \sum_{k=1}^{s-1} k \, [\lambda_{k+1}^{(\rho\mu)(1)} \overline{J_{\Phi,s-k-1}^{(\rho)}(t,k-1)} +$$
$$\lambda_{k+1}^{(\rho\mu)(2)} J_{\Phi,s-k-1}^{(\rho)}(-\bar{t}+2d,k-1)] + K_2 \sum_{k=0}^{s-1} [\lambda_{k+1}^{(\rho\mu)(1)} \overline{J_{\Psi,s-k-1}^{(\rho)}(t,k)} -$$
$$\lambda_{k+1}^{(\rho\mu)(2)} J_{\Psi,s-k-1}^{(\rho)}(-\bar{t}+2d,k)] \qquad (s=2,3,\ldots)$$

$$J_{\Delta,s}^{(\rho)}(t,k) = \frac{1}{2\pi i} \int_L \Delta_s^{(\rho)}(\xi)(\xi-t)^k \, d\xi \qquad (\Delta = \Phi, \Psi;\ s,k = 0,1,\ldots)$$

Equations (5.5) can be solved consecutively for the elements of the series (5.4) using the Mushelishvili method. At each step of the process (at fixed s) the functions $\Phi_s^{(\mu)}(z)$ and $\Psi_s^{(\mu)}(z)$ are obtained from a set of m_α unconnected equations (5.5) corresponding to various values of μ. The dependence of the system (5.5) on the index μ relates to the fact that one cannot pass to the next stage until all equations of the previous stage have been solved. This results from the indeterminacy of the right-hand sides of the equations.

Neglecting the powers of ε higher than fourth, we reduce the solution of the generalized periodic problem to the case of circular holes and $f^{(\mu)}(t) = A_\mu t$ where A_μ is a real number.

$$\Phi^{(\mu)}(z) = \varepsilon^2 \frac{B_\mu}{K_1} \frac{1}{z-d} - \varepsilon^3 \frac{C_\mu}{K_1} \frac{1}{(z-d)^2} + \frac{\varepsilon^4}{K_1} \Big[D_{\mu 2} \frac{1}{(z-d)^3} - 2 \frac{K_2}{K_1} D_{\mu 1} \frac{1}{z-d} \Big] \tag{5.6}$$

$$\Psi^{(\mu)}(z) = \frac{A_\mu}{K_2} \frac{1}{z-d} + \varepsilon^2 \frac{B_\mu}{K_1} \Big[\frac{1}{(z-d)^3} - \frac{1}{z-d} \Big] +$$
$$\varepsilon^3 2 \frac{C_\mu}{K_1} \Big[\frac{1}{(z-d)^2} - \frac{1}{(z-d)^4} \Big] + \varepsilon^4 \Big[\frac{1}{z-d} D_{\mu 1} \Big(\frac{1}{K_1} + \frac{1}{K_2} + \frac{2K_2}{K_1^2} \Big) -$$
$$\frac{1}{(z-d)^3} \Big(\frac{2K_2}{K_1^2} D_{\mu 1} + \frac{3}{K_1} D_{\mu 2} \Big) + \frac{3}{K_1} D_{\mu 2} \frac{1}{(z-d)^5} \Big]$$

$$B_\mu = \sum_{\rho=1}^{m_\alpha} A_\rho (\lambda_2^{(\rho\mu)(1)} + \lambda_2^{(\rho\mu)(2)}), \quad C_\mu = \sum_{\rho=1}^{m_\alpha} A_\rho (\lambda_3^{(\rho\mu)(1)} - \lambda_3^{(\rho\mu)(2)})$$

$$D_{\mu 1} = \sum_{\rho=1}^{m_\alpha} B_\rho (\lambda_2^{(\rho\mu)(1)} + \lambda_2^{(\rho\mu)(2)}), \quad D_{\mu 2} = \sum_{\rho=1}^{m_\alpha} A_\rho (\lambda_4^{(\rho\mu)(1)} + \lambda_4^{(\rho\mu)(2)})$$

The condition that the flexure function is single-valued means, that in the present load variant the numbers c_μ in (5.1) are assumed to be equal to zero.

Substituting the expressions of the type (5.6) into (3.1) and (3.2), we can construct approximate analytic formulas for the functions $\varphi_{\alpha\nu\mu}(z)$ and $\psi_{\alpha\nu\mu}(z)$.

We illustrate the method by considering a generalized periodic problem of flexure of a thin plate with holes, the contours of which are acted upon by uniformly distributed bending moments. We denote the intensity of these moments on the basic contour by M_μ, assume that the Poisson's ratio is equal to $1/3$, $a = 3$ and $d = 1,4$. We must also set $f^{(\mu)}(t) = -1.5\, M_\mu\,(t-d)$, $K_1 = 5$, $K_2 = -1$.

Table 1

Q	$\theta/\pi = 0$	$1/5$	$2/5$	$1/2$	$3/5$	$7/10$	$4/5$	1
Q_{011}	29	19	11	10	10	12	18	32
$Q_{2/7\pi,11}$	42	32	25	23	22	23	30	50
$Q_{4/7\pi,11}$	30	31	30	27	24	22	26	42
$Q_{6/7\pi,11}$	22	31	34	30	25	22	25	38
Q^*_{011}	123	113	100	90	81	79	99	162
Q^*_{021}	93	97	107	116	126	128	111	61

The Table 1 gives the values of $(-10^2 M_\theta / M)$ at various points of the contour L for the loads Q_{011} $(M_1 = M/7)$ and $Q_{\alpha 11}$ $(M_1 = 2M/7, M_2 = 0)$ for $\alpha = 2\pi/7, 4\pi/7, 6\pi/7$. The control relation $M_r(t) = M_1$ is satisifed with an error not exceeding 6%. The results of the computations for the load Q_{011} (normal periodic problem) show good agreement with the data given in the book [8].

6. More general (translation-truncated) case of loading. We assume that the load $Q^*_{\alpha\nu\mu}$ $(\mu = 1, 2, \ldots, m_\alpha)$ is transformed according to the irreducible representation $\tau_{\alpha\nu}{}^*$ of the subgroup $C_s{}^* \subset C_s$. The length of the basic vector of the subgroup $C_s{}^*$ is denoted by $2l^*$, and $l^* = nl$ where n is an integer. The functions $Q^*_{\alpha\nu\mu}$ are defined arbitrarily on the first n contours counted from the basic contour in the direction of the x-axis, and are defined uniquely on the remaining hole contours using expressions of the type (1.1). Such a class of loadings was studied in [3] where an algorithm was given for a finite expansion of the functions $Q^*_{\alpha\nu\mu}$ into terms which are transformable according to irreducible representations of the complete group C_s of symmetry of the medium. (When using the formulas of [3], we must remember the differences in notation, in particular the use of the asterisk). What was said above implies that any problem belonging to the class specified above can be reduced to a finite number of generalized periodic problems.

Let Q denote an arbitrary contour loading, and $Q\,|_m{}^{(0)}\,(t_m{}^{(0)})$ and $Q\,|_m{}^{(1)}\,(t_m{}^{(1)})$ be the corresponding loads on the contours $L_m{}^{(0)}$ and $L_m{}^{(1)}$, i.e.

$$Q\,(t_m^{(0)}) = Q\,\Big|_m^{(0)}\,(t_m^{(0)}), \quad Q\,(t_m^{(1)}) = Q\,\Big|_m^{(1)}\,(t_m^{(1)}) \tag{6.1}$$

Following [3], it is expedient to introduce the notation

$$Q^*_{\alpha\mu} = \begin{cases} Q^*_{\alpha\nu 1}\delta_{\nu\mu} & (\alpha = 0, \pi) \\ Q^*_{\alpha 1\mu} & (0 < |\alpha| < \pi) \end{cases}, \quad Q_{\beta\mu} = \begin{cases} Q_{\beta\mu 1} & (\beta = 0, \pi) \\ Q_{\beta 1\mu} & (0 < |\beta| < \pi) \end{cases} \quad (\mu = 1, 2)$$

where $\delta_{\nu\mu}$ is the Kronecker delta.

If the loads $Q_{\beta\eta}$ $(\eta = 1, 2)$ are related to the expansions of the functions $Q^*_{\alpha\mu}$, then using some of the formulas of [3], relations (1.1) and (6.1) and the elementary properties of the irreducible representations, we can establish the relation

$$Q_{\beta\eta}(t) = \sum_{\mu=1}^{2}\Big[\sum_{m=0}^{n-N-1} \tau_{\beta\mu\eta}(T_m)\, Q^*_{\alpha\mu}\big|_m^{(0)}(t+2ml) - \sum_{m=1}^{N} \tau_{\beta\mu\eta}(\theta_m)\, \overline{Q^*_{\alpha\mu}\big|_m^{(1)}(-\bar{t}+2ml)}\Big] \qquad (6.2)$$

where $\tau_{\beta\mu\eta}(g)$ $(\mu, \eta = 1, 2;\ g \in C_s)$ is the $\mu\eta$-element of the matrix $\tau_\beta(g)$ given in [3], $N = n/2$ and $N = (n-1)/2$ for the even and odd values of n, respectively.

The scheme for computing the stress-strain state of the plate under the load $Q^*_{\alpha\nu\mu}$ is as follows: using the theorem of [3], we determine the structure of the expansion of the function $Q^*_{\alpha\nu\mu}$, i. e. we clarify the generalized periodic problems to be solved; then we use the formulas (6.2) to find the values of the loads on the contour L for these problems; finally we study the problems and apply the principle of superposition.

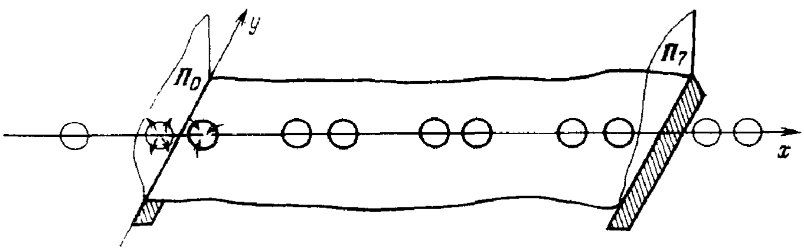

Fig. 2

As an example, let us consider a plate $(a = 3; d = 1{,}4)$ in which the contour of every fourteenth hole, as counted from the basic hole, is acted upon by uniformly distributed bending moments of intensity $2M$. Let us separate the loading of the plate into loads symmetric Q^*_{011} and skew symmetric Q^*_{021} with respect to the imaginary axis.

The symmetric component is transformed according to the representation $\tau_{01}{}^*$ of the group $C_s{}^*$ $(n = 7)$. In accordance with [3] we have

$$Q^*_{011} = Q^*_{01} = \sum_{j=0}^{N} \frac{m_{j\beta}}{n m_\alpha} Q_{j\beta,1} = \frac{1}{7} Q_{011} + \frac{2}{7}(Q_{\beta 11} + Q_{2\beta,11} + Q_{3\beta,11}) \qquad (6.3)$$

$$\left(\beta = \frac{2\pi}{n}\right)$$

Since

$$Q^*_{01}\big|_0^{(0)}(t_0^{(0)}) = M(t-d),\ Q^*_{01}\big|_m^{(0)}(t_m^{(0)}) = Q^*_{01}\big|_m^{(1)}(t_m^{(1)}) = 0 \quad (m = 1, 2, 3)$$

$$Q^*_{02} \equiv 0$$

the formula (6.2) implies that $Q_{j\beta,11}(t) = M(t-d)$ and $Q_{j\beta,12}(t) = 0$ $(\beta = 2\pi/7;$ $j = 0,1,2,3)$. The four generalized periodic problems corresponding to the expansion

(6.3) were solved in Sect. 5. The values of $(-10^2 M_\theta / M)$ on the contour L were given for the load Q_{011}^* in the penultimate row of the table, the latter obtained by summing the upper rows. The last row of the table corresponds to the load Q_{021}^*. The skew symmetric component is transformed according to the representation τ_{02}^* ($n = 7$). We have

$$Q_{021}^* = Q_{02}^* = \sum_{j=0}^{N} \frac{m_{j\beta}}{n m_\alpha} Q_{j\beta,2} = \frac{1}{7} Q_{021} + \frac{2}{7} (Q_{\beta 12} + Q_{2\beta,12} + Q_{3\beta,12})$$
$$\left(\beta = \frac{2\pi}{n}\right)$$

with $Q_{j\beta,12}(t) = M(t-d)$ and $Q_{j\beta,11}(t) = 0$ ($\beta = 2\pi/7$; $j = 0,1,2,3$). We note that in the present case the stress-strain state of the section of the plate contained between the planes Π_0 and Π_7 and of the freely supported strip (Fig. 2) weakened by a transverse set of circular holes, coincide.

REFERENCES

1. Grigoliuk, E.I. and Fil'shtinskii, L.A. Perforated Plates and Shells. Moscow, "Nauka", 1970.

2. Mikhlin, S.G., Integral Equations and their Applications to Certain Problems of Mechanics and Mathematical Physics. Moscow — Leningrad, Gostekhizdat, 1949. (see also English Translation, Pergamon Press, Books No. 10146, and No. 10535, 1964, 1965).

3. Buryshkin, M.L., On static and dynamic computations of one-dimensional regular systems. PMM, Vol. 39, No. 3, 1975.

4. Mushelishvili, N.I., Some Basic Problems of the Mathematical Theory of Elasticity. (English translation), Groningen, Noordhoff, 1953.

5. Vorovich, I.I. and Kosmodamianskii, A.S., Elastic equilibrium of an isotropic plate weakened by a row of identical curvilinear openings. Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie, No. 4, 1959.

6. Cherepanov, G.P., Inverse problems of the plane theory of elasticity. PMM, Vol. 38, No. 6, 1974.

7. Kosmodamianskii, A.S., Stress distribution in isotropic multi-connected media. Izd. Donetsk. gos. Univ. 1972.

8. Kosmodamianskii, A.S. and Ivanov, G.M., Flexure of thin multi-connected plates. Izd. Donetsk. Univ. 1973.

Translated by L. K.